Advanced Data Structures

Lecture 06: BSP Trees and Packed and Compressed Hash Tables

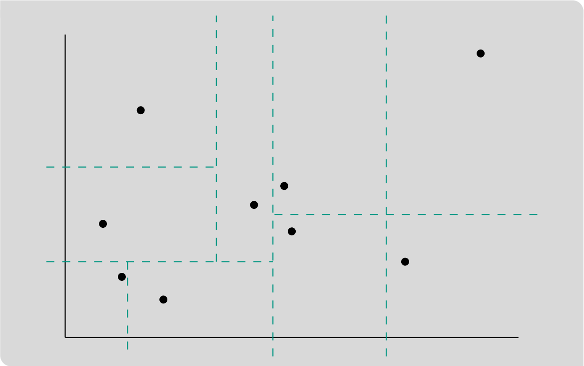
Florian Kurpicz

Recap: 2-Dimensional Rectangular Range Searching

Important

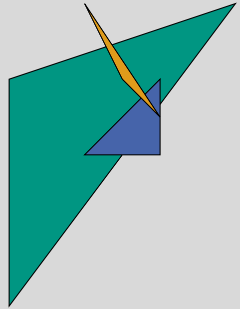
- assume now two points have the same x- or y-coordinate

- generalize 1-dimensional idea
- 1-dimensional
  - split number of points in half at each node
  - points consist of one value
- 2-dimensional
  - points consist of two values
  - split number of points in half w.r.t. one value
  - switch between values depending on depth



Motivation

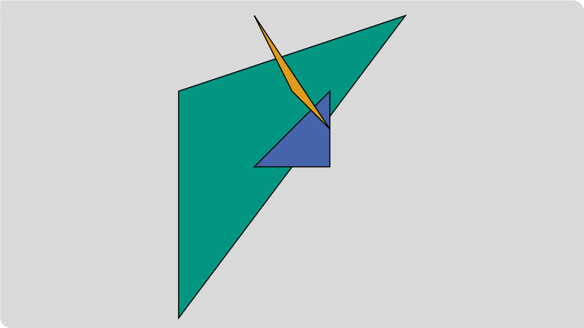
- hidden surface removal
- which pixel is visible
- important for rendering



z-Buffer Algorithm

- transform scene such that viewing direction is positive z-direction
- consider objects in scene in arbitrary order
- maintain two buffers
  - frame buffer ℹ currently shown pixel
  - z-buffer ℹ z-coordinate of object shown
- compare z-coordinate of z-buffer and object

- first sort object in depth-order
- depth-order may not always exist
- how to efficiently sort objects?

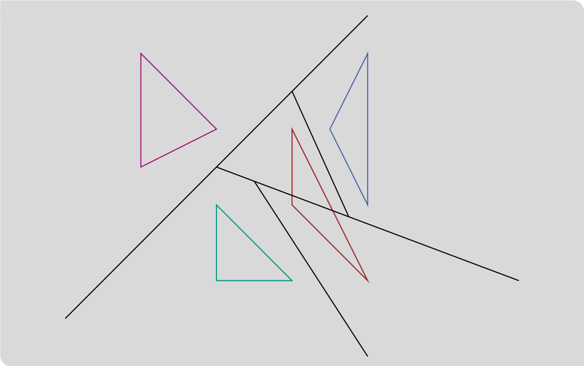


BSP Trees (1/2)

- partition space using hyperplanes
- binary partition ⓘ similar to kd-tree
- hyperplanes create half-spaces and cut objects into fragments

- $h^+ = \{(x_1, \dots, x_d) \colon a_1 x_1 + \dots + a_d x_d > 0\}$
- $h^- = \{(x_1, \dots, x_d) \colon a_1 x_1 + \dots + a_d x_d < 0\}$

- each split creates two nodes in a tree
- if number of objects in space is one: leaf
- otherwise: inner node

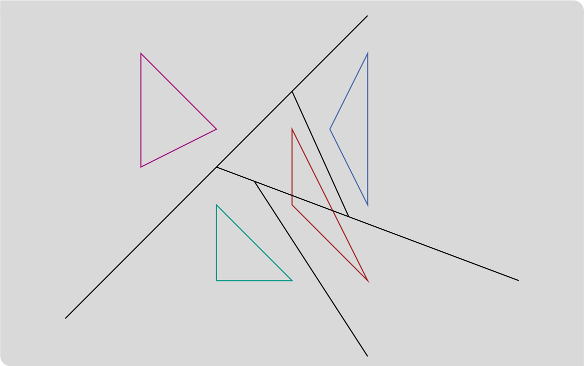


BSP Trees (2/2)

Karlsruhe Institute of Technology

- for leaf: store object/fragment
- for inner node v: store hyperplane h_v and the objects contained in h_v
- left child represents objects in upper half-space h^+
- right child represents objects in lower half-space h^-

- space of BSP tree is number of objects stored at all nodes
- what about fragments?
- too many fragments can make the tree big

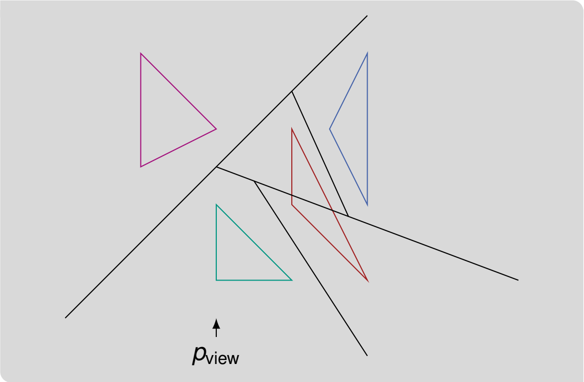


Auto-Partitioning

- sorting points for kd-trees worked well
- BSP-tree is used to sort objects in dept-order
- auto-partitioning uses splitters through objects
  - 2-dimensional: line through line segments
  - 3-dimensional: half-plane through polygons

Painter's Algorithm

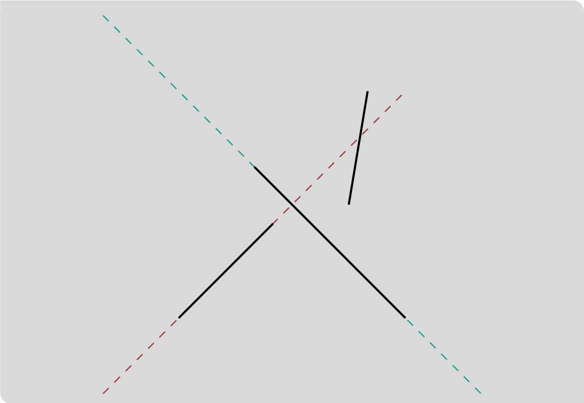
- consider view point p_{view}
- traverse through tree and always recurse on half-space that does not contain p_{view} first
- then scan-convert object contained in node
- then recurse on half-space that contains p_{view}



Constructing Planar BSP Trees (1/3)

- use auto-partitioning
- construction similar to construction of kd-tree
- store all necessary information
  - hyperplane
  - objects in hyperplane
- how to determine next hyperplane?
- creating fragments increases size of BSP tree

- let s be object and $\ell(s)$ line through object
- order matters



Constructing Planar BSP Trees (2/3)

Lemma: Number Line Fragments

The expected number of fragments generated when iterating through the line segments using a random permutation is $O(n \log n)$

Proof (Sketch)

- distance of lines $dist_{s_i}(s_j) =$
$$\begin{cases} \#\text{ segments inters. } \ell(s_i) & \\ \text{ between } s_i \text{ and } s_j & \ell(s_i) \text{ inters. } s_j \\ \infty & \text{otherwise} \end{cases}$$
- example on the board

Proof (Sketch, cnt.)

- let $dist_{s_i}(s_j) = k$ and $s_{j_1}, \ldots, s_{j_k}$ be segments between s_i and s_j
- what is the probability that $\ell(s_i)$ cuts s_j?
- this happens if no s_{j_x} is processed before s_i
- since order is random

$$\mathbb{P}[\ell(s_i) \text{ cuts } s_j] \leq \frac{1}{dist_{s_i}(s_j) + 2}$$

Constructing Planar BSP Trees (3/3)

Proof (Sketch, cnt.)

- expected number of cuts

$$\mathbb{E}[\#\text{ cuts generated by } s_i] \leq \sum_{j \neq i} \frac{1}{dist_{s_i}(s_j) + 2} \leq 2 \sum_{k=0}^{n-2} \frac{1}{k+2} \leq 2 \ln n$$

- all lines generate at most $2n \ln n$ fragments

Lemma: BSP Construction

A BSP tree of size $O(n \log n)$ can be computed in expected time $O(n^2 \log n)$

Proof (Sketch)

- computing permutation in linear time
- construction is linear in number of fragments to be considered
- number of fragments in subtree is bounded by n
- number of recursions is $n \log n$

New Topic: Hash Tables

- now hash tables
- first packed and compressed hash table
- presented in January ’23 at ALENEX

Motivation

Setting

- static hash table for objects of variable size
- storing objects in external memory
- ideally retrieve objects in single I/O
- very small internal memory data structure

Objects of Variable Size

External Memory

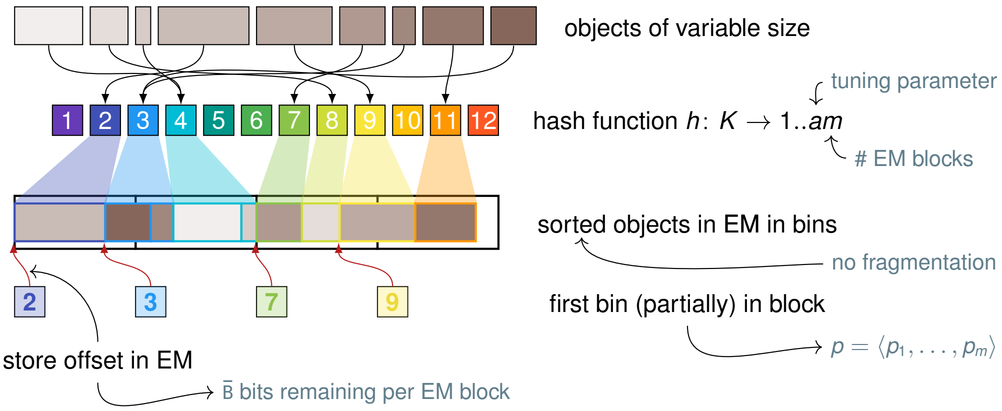
- only blocks of size B bits can be transferred
- one I/O per block transfer

Space-Efficient Object Stores from Literature

- objects of size 256 bytes
- blocks of size 4096 bytes
- internal space I_b (bits/block)
- (*) consecutive I/O

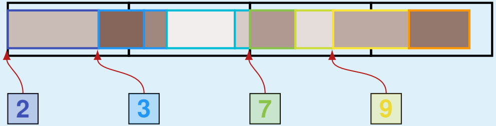
	Method	I_b	load factor	I/Os
fixed	Larson et al. [LR85]	96	$<$96 %	1
	SILT SortedStore [Lim+11]	51	100 %	1
	Linear Separator [Lar88]	8	85 %	1
	Separator [GL88; LK84]	6	98 %	1
	Robin Hood [Cel88]	3	99 %	1.3
	Ramakrishna et al. [RT89]	4	80 %	1
	Jensen, Pagh [JP08]	0	80 %	1.25
	Cuckoo [Aza+94; Pag03]	0	$<$100 %	2
	PaCHash, $a = 1$	2	100 %	2*
	PaCHash, $a = 8$	5	100 %	1.13*
variable	SILT LogStore [Lim+11]	832	100 %	1
	SkimpyStash [DSL11]	32	$\leq$98 %	8
	PaCHash, $a = 1$	2	99.95 %	2.06*
	PaCHash, $a = 8$	5	99.95 %	1.19*

PaCHash Overview



Finding Blocks

Query Algorithm



- $b_x = h(x)$
- find first i with $p_i \leq b_x$
- if $p_i = b_x$ let $i = i - 1$
- find first j with $p_j > b_x$
- return $i..(j-1)$

Elias-Fano Coding

- given k monotonic increasing integers in $1..u$
  - store $\log k$ MSBs encoded in bit vector
  - store $\log(u/k)$ LSBs plain
  - $k(2 + \log(u/k)) + 1 + o(k)$ bits in total
- predecessor in $O(k)$ time

Lemma: Space with Elias-Fano Coding

When using Elias-Fano coding [Eli74; Fan71] to store p, the index needs $2 + \log a + o(1)$ bits of internal memory per block.

Predecessor Query in PaCHash Internal Memory

Karlsruhe Institute of Technology

Lemma: Expected Predecessor Time

When using Elias-Fano coding to store p, the range of blocks containing the bin of an object x can be found in expected constant time.

Proof (Sketch)

- consider $\lceil \log m \rceil$ MSB
- let bin b_x have MSBs equal to u
- expected size $\mathbb{E}(Y_u)$ of all bins with MSB u that are $< b_x$ is

$$\sum_{y \in S} |y| \cdot \mathbb{P}(h(y) \text{ w/ MSB} = u; h(y) < h(x))$$

$$\leq \sum_{y \in S} |y| \cdot \mathbb{P}(h(y) \text{ w/ MSB} = u)$$

$$= \frac{1}{m} \sum_{y \in S} |y| = \frac{m\bar{\mathrm{B}}}{m} = \bar{\mathrm{B}}$$

- number of entries to scan is $\mathbb{E}(Y_u)/\bar{\mathrm{B}} = 1$

Loading Blocks from External Memory

Karlsruhe Institute of Technology

Lemma: Additional Blocks Loaded

Retrieving an object x of size $|x|$ from a PaCHash data structure loads $\leq 1 + |x|/\bar{B} + 1/a$ consecutive blocks from the external memory in expectation.

Proof (Sketch)

- expected size of bin $b_x = h(x)$

$$\mathbb{E}(|b_x|) = |x| + \sum_{y \in S, y \neq x} |y| \mathbb{P}(y \in b_x)$$

$$\leq |x| + \sum_{y \in S} |y| \mathbb{P}(y \in b_x)$$

$$= |x| + \sum_{y \in S} |y| \cdot \frac{1}{am} = |x| + \frac{\bar{B}}{a}$$

Proof (Sketch, cnt.)

- expected number of blocks overlapped by b_x

$$\mathbb{E}(X) = 1 + (\mathbb{E}(|b_x|) - 1)/\bar{B}$$

$$= 1 + \frac{|x|}{\bar{B}} + \frac{1}{a} - 1/\bar{B}$$

- $\mathbb{P}(\text{bin and block border align}) = 1/\bar{B}$

Experimental Evaluation

Hardware and Software

- Intel i7 11700 (base clock speed: 2.5 GHz)
- 1 TB Samsung 980 Pro NVMe SSD
- Ubuntu 21.10 (Kernel 5.13.0)
- `io_uring` for I/O operations
- GCC 11.2.0 (`-O3 -march=native`)
- $B = 4096$ bytes

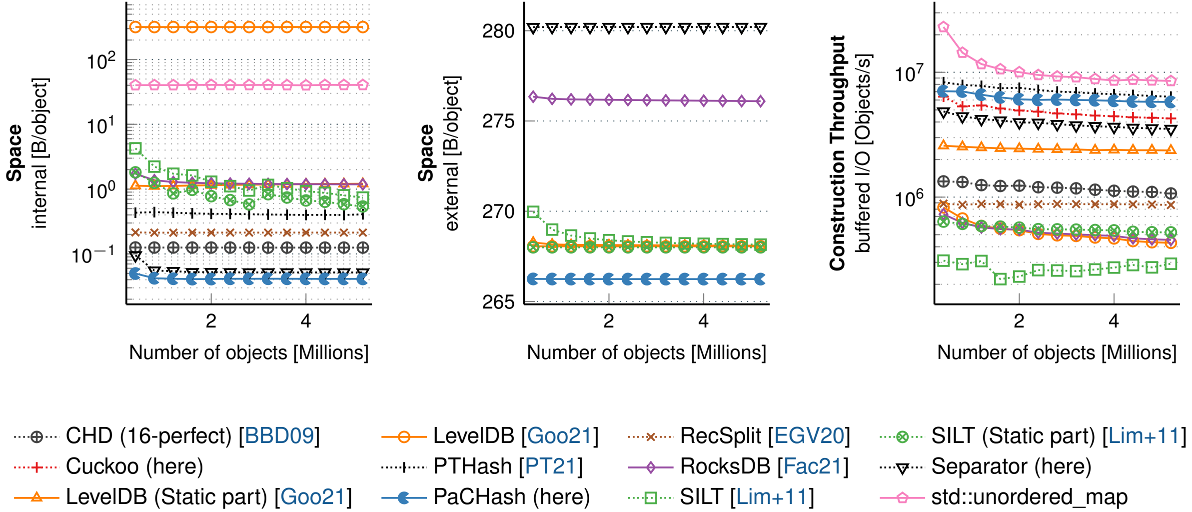
Objects

- here only **fixed size**
- more in the paper (very similar results)

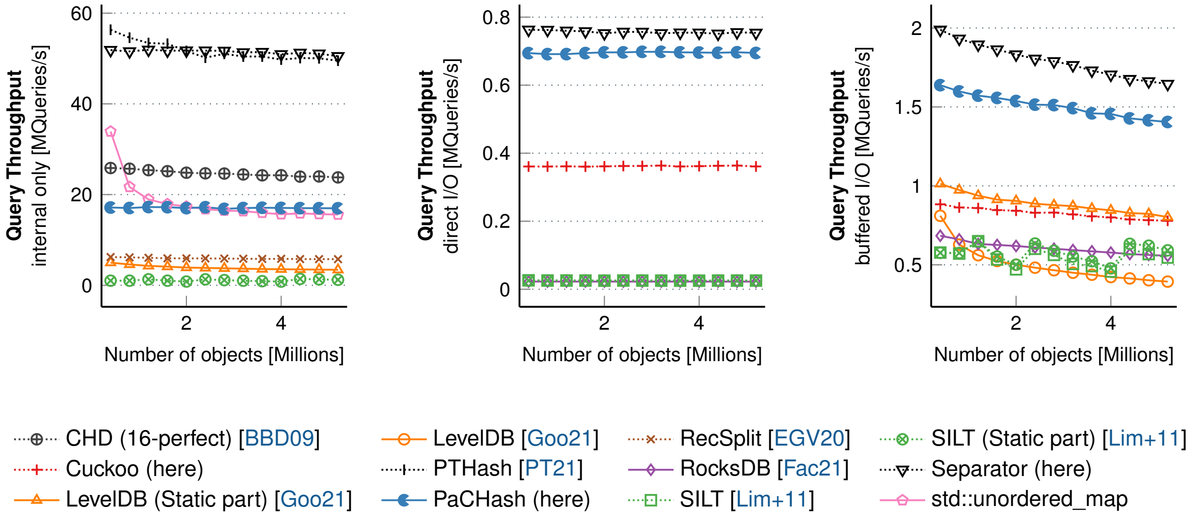
Competitors

- LevelDB [Goo21]
- RocksDB [Fac21]
- SILT [Lim+11].
- `std::unordered_map`
- RecSplit [EGV20]
- CHD [BBD09; CR+12]
- PTHash [PT21]

Construction

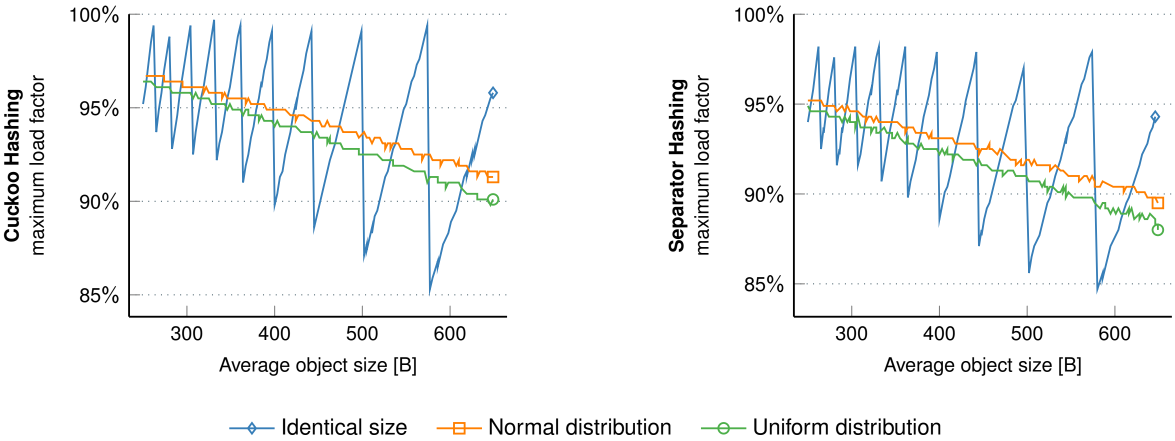


Queries



Query Throughput
internal only [MQueries/s]
60
40
20
0
2
4
Number of objects [Millions]
Query Throughput
direct I/O [MQueries/s]
0.8
0.6
0.4
0.2
0
2
4
Number of objects [Millions]
Query Throughput
buffered I/O [MQueries/s]
2
1.5
1
0.5
2
4
Number of objects [Millions]
CHD (16-perfect) [BBD09]
Cuckoo (here)
LevelDB (Static part) [Goo21]
LevelDB [Goo21]
PTHash [PT21]
PaCHash (here)
RecSplit [EGV20]
RocksDB [Fac21]
SILT [Lim+11]
SILT (Static part) [Lim+11]
Separator (here)
std::unordered_map

Maximum Load Factor of Competitors



Alternative Internal Memory Data Structures

Lemma: Space with Succincter

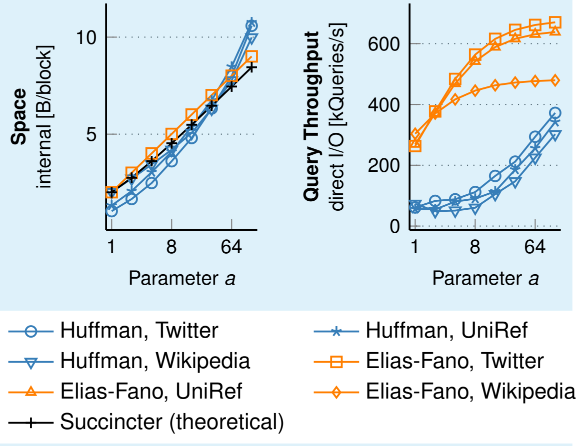
When using Succincter [Pat08] to store p, the index needs $1.44 + \log(a+1) + o(1)$ bits of internal memory per block.

Structure of Bit Vector

- runs of `0`s and `10`s
- sometimes additional `1`s

Entropy Encoding

- encode positions directly
- compress bit vector using Huffman codes
- encode blocks of size 8, 16, 32, or 64

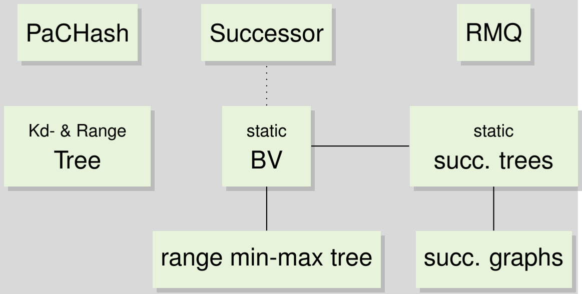


Conclusion and Outlook

This Lecture
- BSP trees
- PaCHash

Next Lecture
- more on hashing

Advanced Data Structures



F.A.Q. Project

- measuring memory
- measuring time
- `std::map` vs `st::unordered_map`
- more questions?

Bibliography I

[Aza+94] Yossi Azar, Andrei Z. Broder, Anna R. Karlin, and Eli Upfal. "Balanced allocations (extended abstract)". In: *STOC*. ACM, 1994, pages 593–602. DOI: 10.1145/195058.195412.

[BBD09] Djamal Belazzougui, Fabiano C. Botelho, and Martin Dietzfelbinger. "Hash, Displace, and Compress". In: *ESA*. Volume 5757. Lecture Notes in Computer Science. Springer, 2009, pages 682–693. DOI: 10.1007/978-3-642-04128-0_61.

[Cel88] Pedro Celia. *External Robin Hood Hashing*. Technical report. Computer Science Department, Indiana University. TR246, 1988.

[CR+12] Davi de Castro Reis, Djamel Belazzougui, Fabiano Cupertino Botelho, and Nivio Ziviani. *CMPH - C Minimal Perfect Hashing Library*. http://cmph.sourceforge.net/. 2012.

[DSL11] Biplob K. Debnath, Sudipta Sengupta, and Jin Li. "SkimpyStash: RAM space skimpy key-value store on flash-based storage". In: *SIGMOD Conference*. ACM, 2011, pages 25–36. DOI: 10.1145/1989323.1989327.

Bibliography II

[EGV20] Emmanuel Esposito, Thomas Mueller Graf, and Sebastiano Vigna. "RecSplit: Minimal Perfect Hashing via Recursive Splitting". In: *ALENEX*. SIAM, 2020, pages 175–185. DOI: 10.1137/1.9781611976007.14.

[Eli74] Peter Elias. "Efficient Storage and Retrieval by Content and Address of Static Files". In: *J. ACM* 21.2 (1974), pages 246–260. DOI: 10.1145/321812.321820.

[Fac21] Facebook. *RocksDB. A Persistent Key-Value Store for Fast Storage Environments*. https://rocksdb.org. 2021.

[Fan71] Robert Mario Fano. *On the number of bits required to implement an associative memory*. Technical report. Project MAC, Memorandum 61". MIT, Computer Structures Group, 1971.

[GL88] Gaston H. Gonnet and Per-Åke Larson. "External hashing with limited internal storage". In: *J. ACM* 35.1 (1988), pages 161–184. DOI: 10.1145/42267.42274.

[Goo21] Google. *LevelDB is a Fast Key-Value Storage Library Written at Google*. https://github.com/google/leveldb. 2021.

Bibliography III

[JP08] Morten Skaarup Jensen and Rasmus Pagh. "Optimality in External Memory Hashing". In: *Algorithmica* 52.3 (2008), pages 403–411. DOI: 10.1007/s00453-007-9155-x.

[Lar88] Per-Åke Larson. "Linear Hashing with Separators - A Dynamic Hashing Scheme Achieving One-Access Retrieval". In: *ACM Trans. Database Syst.* 13.3 (1988), pages 366–388. DOI: 10.1145/44498.44500.

[Lim+11] Hyeontaek Lim, Bin Fan, David G. Andersen, and Michael Kaminsky. "SILT: a memory-efficient, high-performance key-value store". In: *SOSP*. ACM, 2011, pages 1–13. DOI: 10.1145/2043556.2043558.

[LK84] Per-Åke Larson and Ajay Kajla. "File Organization: Implementation of a Method Guaranteeing Retrieval in One Access". In: *Commun. ACM* 27.7 (1984), pages 670–677. DOI: 10.1145/358105.358193.

[LR85] Per-Åke Larson and M. V. Ramakrishna. "External Perfect Hashing". In: *SIGMOD Conference*. ACM Press, 1985, pages 190–200. DOI: 10.1145/318898.318916.

Bibliography IV

[Pag03] Rasmus Pagh. "Basic External Memory Data Structures". In: *Algorithms for Memory Hierarchies*. Volume 2625. Lecture Notes in Computer Science. Springer, 2003, pages 14–35. DOI: 10.1007/3-540-36574-5_2.

[Pat08] Mihai Patrascu. "Succincter". In: *FOCS*. IEEE Computer Society, 2008, pages 305–313. DOI: 10.1109/FOCS.2008.83.

[PT21] Giulio Ermanno Pibiri and Roberto Trani. "PTHash: Revisiting FCH Minimal Perfect Hashing". In: *SIGIR*. ACM, 2021, pages 1339–1348. DOI: 10.1145/3404835.3462849.

[RT89] M. V. Ramakrishna and Walid R. Tout. "Dynamic External Hashing with Guaranteed Single Access Retrieval". In: *FODO*. Volume 367. Lecture Notes in Computer Science. Springer, 1989, pages 187–201. DOI: 10.1007/3-540-51295-0_127.